

This is how *Rabainu Tahm* - הכי גרס רבינו תם פש להו תרי תילתא אמתא - reads the text; ‘there still remains two-thirds of an *Ahmoh*’

Overview

עולא maintains that a tree is יונק from an circular area with a radius of 16 אמות around the tree. This is an area of 768 square אמות.¹ From the משנה on כז,ב regarding three trees in a בית סאה (which is 2500 square אמות) it turns out that each tree is יונק an area of 833 and 1/3 square אמות. We have here a discrepancy of 65 and 1/3 square אמות, which עולא is less than the משנה. In order to make the measurement of עולא (nearly) equal to the משנה, we will need to increase the radius to somewhat more than 16. The גמרא in our גירסא is that we are missing one-half אמה (meaning that the radius should be 16.5 אמות), while according to the ר"ת it should be increased 2/3 of an אמה to 16 and 2/3).²

פירוש כשתקיף אילן דעולא רצועה רחבה שני שלישי אמה³ -

The explanation of אמתא (we are missing 2/3 of an אמה) is that when you will surround the tree of עולא (which has a radius of 16 אמה) with a strip of land which is 2/3 of an אמה wide, meaning that the radius will now be 16 and 2/3 אמה, this new area of יניקה -

יהא גדול כאילן המשנה שהיה יותר ס"ה אמות ושליש ברוחב אמה⁴ -

Will be as large as the tree of the משנה whose area was larger by 65 and 1/3 אמה with a width of an אמה (meaning 65 and 1/3 sq. אמות) –

will now show how adding the additional 2/3 אמות, will make s'עולא tree יונק from the same area as the משנה tree:

¹ The גמרא reaches this conclusion by first squaring the area (which is 32 x 32 אמות [twice the radius of 16]) which is 1024 square אמות. We then reduce the area of a circle from the area of a square by 1/4 (which is 256); subtract that from 1024 it equals 768. Using the equation of $A = \pi r^2$ (where $\pi = 3$ as the גמרא assumes), we achieve the same result; $3 \times 16 \times 16 = 768$. [A is the area of the circle; π is the symbol for pi which is the ratio of the circumference of a circle to its diameter. This is not an exact number; the גמרא assumes it to be 3, and r refers to the radius of the circle.]

² Using the same formula as in footnote # 1 if $r = 16.5$ (as in our גירסא) the area will be 816.75 (still a difference of 16.58 sq. אמות approximately, from the משנה); if, however, $r = 16.66...$ (as the ר"ת would have it) the area would be 833.33; virtually the same as the משנה. This is why the ר"ת prefers his גירסא. However, it seems that they did not use this formula (in footnote # 1) in those days, therefore it was necessary for תוספות to use other methods to achieve his goal of finding the right amount that the radius needs to be increased, in order to match עולא with the משנה.

³ This means literally that if you will add a strip of land 2/3 of an אמה outside the circumference of the circle which has a radius of sixteen אמות from the tree (making the new radius 16 2/3 אמות).

⁴ See ‘Overview’ and in the גמרא.

שהרי אם תקיף ל"ב על ל"ב מרובעים⁵ יצטרך קכ"ח⁶ ולד' קרנות ח' שלישים⁷ -

For if you will surround a square of 32 by 32 with a strip of $\frac{2}{3}$ אמה width, it will require a strip of 128 אמה long and for the four corners an additional $\frac{8}{3}$ אמה -

הרי ק"ל ושני שלישים⁸ אורך על שני שלישים רוחב -

So now we have a strip of 130 and $\frac{2}{3}$ אמות long by $\frac{2}{3}$ אמה wide; this is the extra area added on by adding $\frac{2}{3}$ אמה around the entire square. However, this is by a square, but we are dealing with a circle, therefore -

תחסר הרביעי⁹ שהעיגול פחות ישאר צ"ח -

Diminish this amount by a fourth, which is the amount the area of a circle is less than a square (who share the same diameter) there will remain a strip of 98 by $\frac{2}{3}$ אמה; we now have the area we have added to עולא, by adding the strip of $\frac{2}{3}$ אמה, namely $98 \times \frac{2}{3}$ אמה. This will equal the $65 \frac{1}{3}$ אמה which the משנה was greater than עולא, as תוספות will now show -

וכן ס"ה אמה ושליש תחתוך לג' חלקים לאורך תהיה רצועה אחת של קצ"ו¹⁰ ברוחב שלישי -

And similarly, the 65 and $\frac{1}{3}$ אמה (which is the amount the area of the משנה was greater than the original area of עולא), take this strip and cut it in the length into three equal parts and put them next to each other in length, you will have one strip of 196 אמה long by $\frac{1}{3}$ אמה wide -

תחתוך לשנים ותשים זה בצד זה תהיה רצועה ארכה צ"ח על רוחב שני שלישי¹¹ אמה¹² -

Cut this long strip of 196 אמה into two equal parts (each one 98 אמה) there will be a strip of 98 אמה long by a width of $\frac{2}{3}$ אמה.

אילן דמשנה offers an alternate method to reconcile the אילן דעולא with the תוספות:

עוד מצא רבינו יצחק בספר¹³ אחר מפורש בגמרת הספר להקיף בחוטים¹⁴ אילן דעולא -

⁵ We begin by assuming that the sixteen אמה radius (which is 32 אמה diameter) is a square of 32 אמות (on each side) instead of a circle with a 32 אמה diameter. The way the גמרא calculates the area of a circle is by first assuming a square with the same size as the diameter of the circle, then after calculating the area of the square, they would reduce it by $\frac{1}{4}$ and that would give the area of a circle, so that a circle inscribed in a square 2×2 which has an area of 4 ($2 \times 2 = 4$), the circle (which has a radius of 1), will have an area of 3, a fourth less than the square.

⁶ The perimeter of a 32×32 square is 128 ($32 \times 4 = 128$). However, we need to add the strip to each of the four corners; each corner will require a strip of $\frac{2}{3} \times \frac{2}{3}$ אמה to complete adding $\frac{2}{3}$ of an אמה to the entire square.

⁷ $\frac{8}{3} = 2$ and $\frac{2}{3}$.

⁸ $128 + \frac{8}{3}$ or $128 + 2$ and $\frac{2}{3} = 130$ and $\frac{2}{3}$.

⁹ $130.66.. \times .75 = 97.99995$ or actually 98. We are using only whole numbers. Alternately $130.66/4 = 32.665$; $130.66 - 32.665 = 98$.

¹⁰ $65 \frac{1}{3} \times 3 = 196$ [$65 \times 3 = 195$, and $\frac{1}{3} \times 3 = 1$; in total 196].

¹¹ The long strip was $\frac{1}{3}$ אמה wide when they are joined in the width it is $\frac{2}{3}$ אמה wide.

¹² This area $98 \times \frac{2}{3}$ is exactly the same as the area of עולא if we add a strip of $\frac{2}{3}$ אמה.

¹³ The אמר אחר מפורש בגמרת הספר (instead of מפורש בגמרת הספר).

¹⁴ The advantage of this explanation over the previous one is that practically one cannot take a strip which is a rectangle $\frac{2}{3}$ אמה wide and place it around a circle since the circle is round and a strip is straight; however, with thin threads one can imagine circling the circle with the threads.

Additionally, the ר"י found in another ספר, expressed in the margin of the ספר, to surround, with threads, the outside circumference of the עולה tree -

שהוא ל"ב על ל"ב עד שיהא¹⁵ שני שלישי אמה סביב¹⁶ -

Which has a diameter of 32 by 32, there should be sufficient threads until the [width of these threads] will be 2/3 of an אמה wide around the entire circumference -

תחתוך החוטין ותפשטם יהיה העליון ארכו ק' שהוא מקיף ל"ג ושליש¹⁷ -

Cut the threads and stretch them out in a straight line, the upper (outside) thread will be 100 אמה long since it encircles a diameter of 33 and 1/3 -

וכל שיש ברחבו טפח יש בהיקפו ג'¹⁸ -

And the rule is that if the diameter is a טפה, the circumference is three טפחים -

והתחתון צ"ו שהוא מקיף ל"ב¹⁹ הרי²⁰ התחתון פחות מן העליון מכל צד ב' אמות²¹ -

And the lower (inner) thread is 96 אמות long, since it surrounds the actual diameter of 32, so the [length] of the inner thread is less than the outer thread 2 אמות on each side -

ותחתוך העודף מצד אחד ותמלא צד אחד²² האורך לצד הקצר²³ תהיה רצועה צ"ח²⁴ -

And cut off the extra threads that protrude from one side of the outer threads, and fill in the other side, the long next to the short so you will have a strip of 98 אמה by 2/3 אמה

וכן הוא ס"ה ושליש -

¹⁵ The הגהות הב"ה amends this to read שני שיהא רוחב החוט (יך) (instead of שני שיהא).

¹⁶ The עולה tree has a diameter of 32 אמה. Mark off the circumference of this diameter, and start encircling this circumference (beginning immediately next to the edge of the circumference) with many threads next to each other, until the width from the most outside thread to the edge of the circumference is 2/3 אמה.

¹⁷ The outer thread was 2/3 אמה beyond the circumference (see footnote # 16), this addition is on both (all) sides of the circle; the new diameter (the diameter of this thread is) is an additional (2 x 2/3 or) 1 and 1/3 אמה, making the total diameter 33 and 1/3. [32 + 1 and 1/3 = 33 1/3].

¹⁸ This is the number used for pi in the גמרא in the formula C = pi x diameter. Therefore since the diameter is 33 and 1/3, so 3 x 33 and 1/3 = 100. [the current number for pi is 3.1415...]

¹⁹ 32 x 3 = 96.

²⁰ The הגהות הב"ה amends this to read הרי אורך התחתון (instead of הרי התחתון).

²¹ It is 4 אמות less (100-96=4). If we line up the threads by their center, we can view the closest (shortest) thread as 2 אמות less on each side than the outside (longest) thread.

²² The רש"ש amends this to read אהר (instead of אהד).

²³ Let us assume that there are five threads ranging in length (from the inside out) at 96,97,98,99,100. When they are lying straight the 100 thread is two אמות longer on each end than the 96 thread and one אמה longer than the 98 thread, (1/2 from the 99 thread and 1 1/2 from the 97 thread). When we cut the five threads at one end, there are two אמות from the 100 thread and 1 אמה from the 98 thread (1/2 and 1 1/2 from the 97 and 99 threads respectively). We then take these threads flip them and place them on the other side The longer cut piece to the shorter thread, so that the 2 אמות from the 100 thread goes on the 96 thread making it 98 and the one אמה from the 98 thread (which now became a 97 thread) goes on the original 98 thread and it remains 98, etc.; in short all 5 threads become 98 אמה threads.

²⁴ 98 by 2/3 = 65 1/3 x 1 אמה. [98x2=196; 196/3 = 65 and 1/3]

Which is the same as 65 and 1/3 by one אמה –

Another way of equalizing אילן דעולא with the דמשנה:

דרך אחרת ל"ג ושליש על ל"ג ושליש²⁵ מרובעים עולה -

Another way; 33 and 1/3 by 33 and 1/3 squared, equals -

אלף ומאה וי"א אמות ושליש של שלישי שהיא תשיעית אמה²⁶ -

Eleven hundred eleven sq. אמות plus a third of a third which is a ninth of an אמה -

תחסר הרביעית ישאר תתל"ג ושליש²⁷ הרי אילן המשנה ל"ג ושליש על ל"ג ושליש -

Subtract a fourth (since we are discussing a circle) **the remainder is 833 and 1/3 אמה, which is the equivalent of the tree, which is 33 and 1/3 by 33 and 1/3.**

Another calculation:

(הג"ה. או כעין זה שלישי סאה²⁸ הוי י"ו וב' שלישים על חמשים²⁹ -

(A gloss: Or something like this, a third of a סאה is 16 and 2/3 by 50 -

חתוך באמצע ושים זה על זה יהיה עשרים וחמש על שלשים ושלוש ושליש³⁰) -

Cut the fifty אמה length in the middle and place one next to the other you will have 25 by 33 and 1/3).

Another calculation:

דרך אחרת בית סאה היא מאה על כ"ה³¹ תקח השליש לאילן אחד יהיה ל"ג ושליש על כ"ה -

Another way; a בית סאה is 100 by 25, take one third for one tree, his area will be 33 and 1/3 by 25 in a rectangle -

והם ל"ג ושליש על ל"ג ושליש עגולים -

Which is the equivalent of a circle with a diameter of 33 and 1/3; תוספות shows how -

שהרי מוסיף הרביע מלבר שהיא שלישי מלגיו³² -

Since we add to the area of a circle a fourth from the outside which is a third from the inside -

דהיינו ח' אמה ושליש³³ באורך ל"ג ושליש יהיו ל"ג ושליש על ל"ג ושליש מרובעים -

²⁵ We are adding 2/3 אמה to the length and width of the original 32x32. This comes to 33 1/3.

²⁶ $33 \frac{1}{3} \times 33 \frac{1}{3} = 100/3 \times 100/3 = 10,000/3 = 1111 \frac{1}{9}$.

²⁷ $1111.11 \times .75 = 833.33$.

²⁸ In the משנה there are three trees in a סאה, which is 50x50 אמה.

²⁹ $50/3 = 16 \frac{2}{3}$.

³⁰ $25 \times 33.333 = 833.325$.

³¹ A בית סאה is 2500 sq. אמה which is 50 by 50, or 100 by 25.

³² A circle inscribed in a square that has an area of four, will have an area of three. The area of the square is 'one' more than the circle. This 'one' can be considered a third of the area of the circle (מלגיו), or it can be considered a fourth of the entire area of the square (מלבר). We reduce the area of a square by a fourth to get the area of an inscribed circle and we increase the area of the inscribed circle by one third to get the area of the square

³³ A third מלגיו of 25 is 8 1/3.

Which is 8 and 1/3, to a length of 33 and 1/3, there will then be 33 and 1/3 squared -

(ולספרים דגרסינן פש להו פלגא דאמתא³⁴) -

ועשה אותו עגול שיחסר הרביע שהוספת³⁵ ויהיה עגול ל"ג ושליש על ל"ג ושליש--

And make this round, meaning subtract the fourth which we added, so it will be a circle of 33 and 1/3 by 33 and 1/3.

concluded explaining the גמרא according to his גירסא of אמתא תרי תלתא אמתא, now תוספות will explain the גמרא according to the גירסא of אמתא פלגא דאמתא:

ולספרים דגרסי פש ליה פלגא דאמתא יש לפרש דחשיב עולא י"ו לכל צד לבד מקום האילן -

And according to the text which are גורס, 'there is missing half an אמה', we can explain that when עולא counts the sixteen אמות in all directions, that is excluding the place the tree occupies -

ומקום אילן שלישי על שלישי נמצא אילן דעולא ל"ב ושליש על ל"ב ושליש³⁶ -

And the tree occupies a diameter of a third אמה, it turns out that the diameter of אילן דעולא is 32 and 1/3 -

כשתקיף מכל צד פלגא דאמתא יהיה ל"ג ושליש על ל"ג ושליש³⁷ -

So when you will add the extra half אמה (פלגא דאמתא) around the entire circumference you will have a diameter of 33 and 1/3; which is the same as the אילן דמשנה.

Another explanation and גירסא in our גמרא:

ורבינו משולם מפרש דעולא לא חשיב מקום האילן³⁸ ומקום האילן הוי אמה על אמה -

And explains that עולא does not count the place of the tree, however the tree occupies an אמה in diameter (not a third of an אמה as in the previous explanation) -

נמצא דהוי אילן דעולא ל"ג על ל"ג ואם היו מרובעים היה עולה אלף ופ"ט -

It turns out that אילן דעולא has a diameter of 33, and if it were squared it would add up to 1089 sq. אמות -

תחסר הרביע ישאר תתי"ו ושלושה רבעי אמה³⁹ -

Subtract a quarter for the area of a circle with a diameter of 33, there will remain an area of 816 3/4 אמה.

³⁴ The רש"ל deletes the entire parenthesis.

³⁵ We made it into a square (of 33 and 1/3 by adding a fourth מלבר, we will turn it back into a circle by reducing it a fourth, so the area of a circle with a diameter of 33 1/3 is the same that of a square which is 33 1/3 by 25, which is 833 1/3).

³⁶ The extra third is from the space of the tree, which has a diameter of 1/3.

³⁷ The אילן דמשנה (2500/3) is 833 and 1/3, and אילן דעולא (with the extra אמה חצי) is 33 1/3 in diameter whose area is also 833 1/3 (33.3333x33.3333=1111.10889 x.75=833.33).

³⁸ As in the פלגא דאמתא אמתא.

³⁹ 1089 x .75 = 816.75.

ולא גרסינן (והו) [פשו] להו ז' מאה ושתינ ותמניא -

And we are not גורס, 'there remains 768' sq. אמות, rather there remains $816\frac{3}{4}$ אמות -

נמצא דאילן המשנה יתר על של עולא י"ו אמה ושליש ורביע⁴⁰ -

It turns out that the אילן המשנה is larger than דעולא by $16\frac{7}{12}$ אמה -

ופלגא דאמתא קאי אבית סאה שהיא חמשים על חמשים -

And when the גמרא states פלגא דאמתא, it is referring to the בית סאה which is 50×50 -

כשתסיר פלגא דאמתא לאורך וכן לרוחב יעלה החסרון חמשים אמה חסר רביע⁴¹ -

That when you will remove half an אמה in the length and similarly half an אמה in the width, the part subtracted will be $49\frac{3}{4}$ אמה -

דהיינו לג' אילנות ולאילן אחד יעלה י"ו אמה ושליש ורביע⁴² וישאר⁴³ כאילן דעולא:

This amount of $49\frac{3}{4}$ is for three trees but for one tree for one tree it will equal $16\frac{7}{12}$, so the remainder will be the same as אילן דעולא.

Summary

There are various ways of reconciling the אילן דעולא with דמשנה.

Thinking it over

Why was it necessary for תוספות to cite more than one example how to reconcile אילן דעולא with דמשנה?

⁴⁰ $833\frac{1}{3} - 816\frac{3}{4} = 16\frac{7}{12}$. [$\frac{1}{3} = \frac{4}{12} + \frac{1}{4} = \frac{3}{12} = \frac{7}{12}$.]

⁴¹ Let us assume that we first remove a strip half an אמה in the length, so we have a strip of $50 \times \frac{1}{2}$. Then we remove a strip in the width. However since we already removed $\frac{1}{2}$ אמה, the strip of the width now is only $49\frac{1}{2} \times \frac{1}{2}$. We will add $\frac{1}{4}$ אמה to the $49\frac{1}{2}$ from the 50 so we have two 49.75 strips each $\frac{1}{2}$ אמה wide. Meaning we have a strip 49.75×1 אמה.

⁴² $49.75/3 = 16\frac{7}{12}$. This is the amount that אילן דמשנה was greater than אילן דעולא (see [text by] footnote # 38).

⁴³ After subtracting 49.75 from a בית סאה (2500 square אמות) we have left 2450.25. That is the space required for 3 trees. Divide that by 3 ($2450.25/3$) = 816.75. The same amount אילן דעולא. See footnote # 39.